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Finding symmetry identities for the 2-variable Apostol type polynomials

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Abstract

The main aim of this article is to established certain symmetry identities for the 2-variable Apostol type polynomials. The symmetry identities for some special polynomials related to the 2-variable Apostol type polynomials are deduced as special cases. Certain interesting examples are considered to establish the symmetry identities for the 2-variable Gould-Hopper-Apostol type, 2-variable generalized Laguerre-Apostol type and 2-variable truncated exponential-Apostol type polynomials. The special cases of the symmetry identities associated with these polynomials are also given.

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1 Introduction and preliminaries

The special polynomials of two variables are useful from the point of view of applications in physics. Also, these polynomials allow the derivation of a number of useful identities in a fairly straight forward way and help in introducing new families of special polynomials. For example, Bretti *et al.* [3] introduced general classes of the Appell polynomials of two variables by using properties of an iterated isomorphism, related to the Laguerre-type exponentials. The 2-variable forms of the Hermite, Laguerre and truncated exponential polynomials as well as their generalizations are considered by several authors, see for example [2, 4–7, 9].

In order to further stress the importance of the 2-variable special polynomials, Subuhi Khan and Nusrat Raza [11] considered the 2-variable general polynomials (2VGP) $p_n(x,y)$, which are defined by the generating function of the form:

$$e^{xt}\varphi(y,t)=\sum_{n=0}^{\infty}p_n(x,y)\frac{t^n}{n!},\ p_0(x,y)=1, \tag{1.1}$$

where $\varphi(y,t)$ has (at least the formal) series expansion

$$\varphi(y,t)=\sum_{n=0}^{\infty}\varphi_n(y)\frac{t^n}{n!},\ \varphi_0(y)\neq 0. \tag{1.2}$$

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The 2VGP family $p_n(x,y)$ contains a number of important special polynomials of two variables. Generating functions and series definitions for certain members belonging to the 2VGP family are given in Table 1.

Table 1. Certain members belonging to the 2VGP $p_n(x,y)$ family.

S.No.	$\varphi(y,t)$	Name of the polynomials	Generating functions	Series definitions
I.	e^{yt^m}	Gould-Hopper polynomials $H_n^{(m)}(x,y)$ [9]	$e^{xt+yt^m} = \sum_{n=0}^{\infty} H_n^{(m)}(x,y)\frac{t^n}{n!}$	$H_n^{(m)}(x,y) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k x^{n-mk}}{k!(n-mk)!}$
II.	e^{yt^2}	2-variable Hermite Kampé de Feriet polynomials $H_n(x,y)$ [2]	$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$	$H_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{k!(n-2k)!}$
III.	$C_0(-yt^m)$	2-variable generalized Laguerre polynomials ${}_mL_n(y,x)$ [6]	$e^{xt}C_0(-yt^m) = \sum_{n=0}^{\infty} {}_mL_n(y,x)\frac{t^n}{n!}$	${}_mL_n(y,x) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k x^{n-mk}}{(k!)^2(n-mk)!}$
IV.	$C_0(yt)$	2-variable Laguerre polynomials $L_n(y,x)$ [4]	$e^{xt}C_0(yt) = \sum_{n=0}^{\infty} L_n(y,x)\frac{t^n}{n!}$	$L_n(y,x) = n! \sum_{k=0}^{n} \frac{(-1)^k y^k x^{n-k}}{(k!)^2(n-k)!}$
V.	$\frac{1}{1-yt^r}$	2-variable truncated exponential polynomials of order r, $e_n^{(r)}(x,y)$ [7]	$\frac{e^{xt}}{1-yt^r} = \sum_{n=0}^{\infty} e_n^{(r)}(x,y)\frac{t^n}{n!}$	$e_n^{(r)}(x,y) = n! \sum_{k=0}^{[\frac{n}{r}]} \frac{y^k x^{n-rk}}{(n-rk)!}$
VI.	$\frac{1}{1-yt^2}$	2-variable truncated exponential ${}_{[2]}e_n(x,y)$ [5]	$\frac{e^{xt}}{1-yt^2} = \sum_{n=0}^{\infty} {}_{[2]}e_n(x,y)\frac{t^n}{n!}$	${}_{[2]}e_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{(n-2k)!}$

Recently, Luo and Srivastava [16] introduced a unified family of the generalized Apostol type polynomials. The Apostol type polynomials (ATP) $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ $(\alpha \in \mathbb{N}, \lambda,\mu,\nu \in \mathbb{C})$ of order α, are defined by the generating function of the form:

$$\left(\frac{2^\mu\ t^\nu}{\lambda e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)\frac{t^n}{n!},\ \ |t| < |\log(-\lambda)|. \tag{1.3}$$

The ATP $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ are viewed as a unification and generalization of certain polynomials. We mention these polynomials in Table 2.

Table 2. Certain special cases of the ATP $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$.

S. No.	Values of the parameters	Relation between the ATP $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ and its special case	Name of the resultant special polynomials	Generating functions
I.	$\lambda \to -\lambda$ $\mu = 0,\ \nu = 1$	$(-1)^\alpha \mathcal{F}_n^{(\alpha)}(x;-\lambda;0,1) = \mathfrak{B}_n^{(\alpha)}(x;\lambda)$	Apostol-Bernoulli polynomials of order α, $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ [15]	$\left(\frac{t}{\lambda e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$ $(\lvert t\rvert < \lvert\log\lambda\rvert)$
II.	$\mu = 1,\ \nu = 0$	$\mathcal{F}_n^{(\alpha)}(x;\lambda;1,0) = \mathfrak{E}_n^{(\alpha)}(x;\lambda)$	Apostol-Euler polynomials of order α, $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ [13]	$\left(\frac{2}{\lambda e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$ $(\lvert t\rvert < \lvert\log(-\lambda)\rvert)$
III.	$\mu = \nu = 1$	$\mathcal{F}_n^{(\alpha)}(x;\lambda;1,1) = \mathcal{G}_n^{(\alpha)}(x;\lambda)$	Apostol-Genocchi polynomials of order α, $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ [14]	$\left(\frac{2t}{\lambda e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$ $(\lvert t\rvert < \lvert\log(-\lambda)\rvert)$

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We note that, for $\lambda = 1$, the polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$, $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ and $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ reduce to the Bernoulli polynomials of order α, $B_n^{(\alpha)}(x)$, Euler polynomials of order α, $E_n^{(\alpha)}(x)$ and Genocchi polynomials of order α, $G_n^{(\alpha)}(x)$, which are defined by means of the following generating functions [8, 20]:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \ |t| < 2\pi, \tag{1.4}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \ |t| < \pi, \tag{1.5}$$

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!}, \ |t| < \pi, \tag{1.6}$$

respectively.

For $\alpha = 1$, the polynomials $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ reduce to the Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$, respectively.

In fact from (Table 2 (I, II and III)) and equations (1.4)-(1.6), we have the following relations:

$$\mathfrak{B}_n^{(\alpha)}(x;1) = B_n^{(\alpha)}(x);\ \mathfrak{E}_n^{(\alpha)}(x;1) = E_n^{(\alpha)}(x);\ \mathcal{G}_n^{(\alpha)}(x;1) = G_n^{(\alpha)}(x). \tag{1.7}$$

Also, we note that

$$B_n^{(1)}(x) = B_n(x);\ E_n^{(1)}(x) = E_n(x);\ G_n^{(1)}(x) = G_n(x),\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.8}$$

Recently, Khan *at el.* [12] introduced the 2-variable Apostol type polynomials (2VATP) of order α, ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ ($\alpha \in \mathbb{N}, \lambda,\mu,\nu \in \mathbb{C}$) as the discrete Apostol type convolution of the 2VGP $p_n(x,y)$. The 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ are defined by the following generating function [12, p.1372(2.1)]:

$$\left(\frac{2^{\mu}\ t^{\nu}}{\lambda e^t+1}\right)^{\alpha} e^{xt}\varphi(y,t) = \sum_{n=0}^{\infty} {}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)\frac{t^n}{n!}, \ |t| < |\log(-\lambda)|. \tag{1.9}$$

By making suitable choice for the function $\varphi(y,t)$ in equation (1.9) and in view of Table 1, the generating function and other results for the corresponding member belonging to the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ are obtained in [12].

Yang in [22] derived symmetry identities for the Bernoulli polynomials of order, α $B_n^{(\alpha)}(x)$. Further, Zhang and Yang in [23] derived certain symmetry identities for the polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ and $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ involving generalized sum of integer powers $\mathcal{S}_k(n;\lambda)$ and generalized sum of alternative integer powers $\mathcal{M}_k(n;\lambda)$. Certain results for the Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda)$ of higher order are obtained [10]. Further, Özarslan [17] also derived some symmetry identities for the unified Hermite-based Apostol polynomials ${}_H\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)$.

We recall the following definitions considered in [23]:

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Definition 1.1. For each integer $k \geq 0$, the sum $S_k(n) = \sum\limits_{i=0}^{n} i^k$ is known as the sum of integer powers or simply the power sum. The exponential generating function for $S_k(n)$ is given as:

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = 1 + e^t + e^{2t} + \ldots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1}. \tag{1.10}$$

Definition 1.2. For an arbitrary real or complex parameter λ, the generalized sum of integer powers $\mathcal{S}_k(n;\lambda)$ is defined by the following generating function:

$$\frac{\lambda e^{(n+1)t} - 1}{\lambda e^t - 1} = \sum_{k=0}^{\infty} \mathcal{S}_k(n;\lambda)\frac{t^k}{k!}. \tag{1.11}$$

From equations (1.10) and (1.11), it follows that

$$\mathcal{S}_k(n;1) = S_k(n). \tag{1.12}$$

Definition 1.3. For each integer $k \geq 0$, the sum $M_k(n) = \sum\limits_{i=0}^{n} (-1)^k i^k$ is known as the sum of alternative integer powers. The exponential generating function for $M_k(n)$ is given as:

$$\sum_{k=0}^{\infty} M_k(n)\frac{t^k}{k!} = 1 - e^t + e^{2t} + \ldots + (-1)^n e^{nt} = \frac{1 - (-e^t)^{(n+1)}}{e^t + 1}. \tag{1.13}$$

Definition 1.4. For an arbitrary real or complex parameter λ, the generalized sum of alternative integer powers $\mathcal{M}_k(n;\lambda)$ is defined by the following generating function:

$$\frac{1 - \lambda(-e^t)^{(n+1)}}{\lambda e^t + 1} = \sum_{k=0}^{\infty} \mathcal{M}_k(n;\lambda)\frac{t^k}{k!}. \tag{1.14}$$

We note that

$$\mathcal{M}_k(n;1) = M_k(n). \tag{1.15}$$

Also, for even n, we have

$$\mathcal{S}_k(n;-\lambda) = \mathcal{M}_k(n;\lambda). \tag{1.16}$$

The importance of the 2-variable forms of the special polynomials in applications and the work of Yang [22], Zhang and Yang [23] and Özarslan [17] on symmetry identities provides motivation to consider symmetry identities for more general families. In this article, symmetry identities for the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ are derived. Further, by considering different members of the 2VGP $p_n(x,y)$, the symmetry identities for certain members belonging to this family are also derived.

2 Symmetry identities

In order to derive the symmetry identity for the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, we prove the following result:

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Theorem 2.1. For all integers $c,\ d > 0$ and $n \geq 0,\ \alpha \geq 1,\ \lambda, \mu, \nu \in \mathbb{C}$, the following symmetry identity for the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x, y; \lambda; \mu, \nu)$ holds true:

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^{\nu+k}\, {}_p\mathcal{F}_{n-k}^{(\alpha)}(dx, dy; \lambda; \mu, \nu) \sum_{l=0}^{k} \binom{k}{l} \mathcal{S}_l(c-1; -\lambda)\, {}_p\mathcal{F}_{k-l}^{(\alpha-1)}(cX, cY; \lambda; \mu, \nu)\\ &= \sum_{k=0}^{n} \binom{n}{k} d^{n-k} c^{\nu+k}\, {}_p\mathcal{F}_{n-k}^{(\alpha)}(cx, cy; \lambda; \mu, \nu) \sum_{l=0}^{k} \binom{k}{l} \mathcal{S}_l(d-1; -\lambda)\, {}_p\mathcal{F}_{k-l}^{(\alpha-1)}(dX, dY; \lambda; \mu, \nu).\end{aligned} \tag{2.1}$$

Proof. Let

$$G(t) := \frac{2^{\mu(2\alpha-1)} t^{\nu(2\alpha-1)} e^{cdxt} \varphi(y, cdt)(\lambda e^{cdt}+1) e^{cdXt} \varphi(Y, cdt)}{(\lambda e^{ct}+1)^{\alpha} (\lambda e^{dt}+1)^{\alpha}}, \tag{2.2}$$

which on rearranging the powers becomes

$$G(t) = \frac{1}{c^{\nu\alpha} d^{\nu(\alpha-1)}} \left(\frac{2^{\mu} c^{\nu} t^{\nu}}{\lambda e^{ct}+1}\right)^{\alpha} e^{cdxt} \varphi(y, cdt) \left(\frac{\lambda e^{cdt}+1}{\lambda e^{dt}+1}\right) \left(\frac{2^{\mu} d^{\nu} t^{\nu}}{\lambda e^{dt}+1}\right)^{\alpha-1} e^{cdXt} \varphi(Y, cdt). \tag{2.3}$$

Since the expression (2.3) for $G(t)$ is symmetric in c and d, therefore we can expand $G(t)$ into series in two ways. First, we consider the following expansion:

$$G(t) = \frac{1}{c^{\nu\alpha} d^{\nu(\alpha-1)}} \left(\frac{2^{\mu} (ct)^{\nu}}{\lambda e^{ct}+1}\right)^{\alpha} e^{cdxt} \varphi(dy, ct) \left(\frac{\lambda e^{cdt}+1}{\lambda e^{dt}+1}\right) \left(\frac{2^{\mu} (dt)^{\nu}}{\lambda e^{dt}+1}\right)^{\alpha-1} e^{cdXt} \varphi(cY, dt). \tag{2.4}$$

Using equations (1.9) and (1.11) in the r.h.s. of equation (2.4), we find

$$\begin{aligned}G(t) = \ &\tfrac{1}{c^{\nu\alpha} d^{\nu(\alpha-1)}} \left(\sum_{n=0}^{\infty} {}_p\mathcal{F}_n^{(\alpha)}(dx, dy; \lambda; \mu, \nu) \tfrac{(ct)^n}{n!}\right) \left(\sum_{l=0}^{\infty} \mathcal{S}_l(c-1; -\lambda) \tfrac{(dt)^l}{l!}\right)\\ &\times \left(\sum_{k=0}^{\infty} {}_p\mathcal{F}_k^{(\alpha-1)}(cX, cY; \lambda; \mu, \nu) \tfrac{(dt)^k}{k!}\right),\end{aligned} \tag{2.5}$$

which on using [21, p.890 Corollary 2] gives

$$\begin{aligned}G(t) = \ &\tfrac{1}{c^{\nu\alpha} d^{\nu\alpha}} \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \tbinom{n}{k} c^{n-k} d^{\nu+k}\, {}_p\mathcal{F}_{n-k}^{(\alpha)}(dx, dy; \lambda; \mu, \nu) \sum_{l=0}^{k} \tbinom{k}{l} \mathcal{S}_l(c-1; -\lambda)\\ &\times {}_p\mathcal{F}_{k-l}^{(\alpha-1)}(cX, cY; \lambda; \mu, \nu)\Big) \tfrac{t^n}{n!}.\end{aligned} \tag{2.6}$$

In view of symmetry of c and d in expression (2.2), we have another expansion of $G(t)$ as:

$$G(t) = \frac{1}{d^{\nu\alpha} c^{\nu(\alpha-1)}} \left(\frac{2^{\mu} (dt)^{\nu}}{\lambda e^{dt}+1}\right)^{\alpha} e^{cdxt} \varphi(cy, dt) \left(\frac{\lambda e^{cdt}+1}{\lambda e^{ct}+1}\right) \left(\frac{2^{\mu} (ct)^{\nu}}{\lambda e^{ct}+1}\right)^{\alpha-1} e^{cdXt} \varphi(dY, ct), \tag{2.7}$$

which on using equations (1.9) and (1.11) in the r.h.s. gives

$$\begin{aligned}G(t) = \ &\tfrac{1}{d^{\nu\alpha} c^{\nu(\alpha-1)}} \left(\sum_{n=0}^{\infty} {}_p\mathcal{F}_n^{(\alpha)}(cx, cy; \lambda; \mu, \nu) \tfrac{(dt)^n}{n!}\right) \left(\sum_{l=0}^{\infty} \mathcal{S}_l(d-1; -\lambda) \tfrac{(ct)^l}{l!}\right)\\ &\times \left(\sum_{k=0}^{\infty} {}_p\mathcal{F}_k^{(\alpha-1)}(dX, dY; \lambda; \mu, \nu) \tfrac{(ct)^k}{k!}\right).\end{aligned} \tag{2.8}$$

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Consequently, we have

$$G(t) = \frac{1}{d^{\nu\alpha}c^{\nu\alpha}} \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \binom{n}{k} d^{n-k} c^{\nu+k} {}_p\mathcal{F}_{n-k}^{(\alpha)}(cx, cy; \lambda; \mu, \nu) \sum_{l=0}^{k} \binom{k}{l} \mathcal{S}_l(d-1; -\lambda) \times {}_p\mathcal{F}_{k-l}^{(\alpha-1)}(dX, dY; \lambda; \mu, \nu) \Big) \frac{t^n}{n!}. \tag{2.9}$$

Equating the coefficients of same powers of t in r.h.s. of expansions (2.6) and (2.9), we are led to assertion (2.1). Q.E.D.

Next, we establish another symmetry identity for the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x, y; \lambda; \mu, \nu)$ by proving the following result:

Theorem 2.2. For each pair of positive integers c, d and for all integers $n \geq 0$, $\alpha \geq 1$, $\lambda, \mu, \nu \in \mathbb{C}$, the following symmetry identity for the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x, y; \lambda; \mu, \nu)$ holds true:

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} (-\lambda)^{i+j} c^k d^{n-k} {}_p\mathcal{F}_k^{(\alpha)}\Big(dx + \tfrac{d}{c}i, dy; \lambda; \mu, \nu\Big) {}_p\mathcal{F}_{n-k}^{(\alpha)}\Big(cX + \tfrac{c}{d}j, cY; \lambda; \mu, \nu\Big) \\ = &\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} (-\lambda)^{i+j} d^k c^{n-k} {}_p\mathcal{F}_k^{(\alpha)}\Big(cx + \tfrac{c}{d}i, cy; \lambda; \mu, \nu\Big) {}_p\mathcal{F}_{n-k}^{(\alpha)}\Big(dX + \tfrac{d}{c}j, dY; \lambda; \mu, \nu\Big). \end{aligned} \tag{2.10}$$

Proof. Let

$$H(t) := \frac{2^{2\mu\alpha} t^{2\nu\alpha} e^{cdxt} \varphi(y, cdt)(\lambda^c e^{cdt} + 1)(\lambda^d e^{cdt} + 1) e^{cdXt} \varphi(Y, cdt)}{(\lambda e^{ct} + 1)^{\alpha+1} (\lambda e^{dt} + 1)^{\alpha+1}}, \tag{2.11}$$

which on rearranging the powers becomes

$$H(t) = \frac{1}{c^{\nu\alpha} d^{\nu\alpha}} \left(\frac{2^\mu c^\nu t^\nu}{\lambda e^{ct} + 1}\right)^{\alpha} e^{cdxt} \varphi(y, cdt) \Big(\frac{\lambda^c e^{cdt} + 1}{\lambda e^{dt} + 1}\Big) \left(\frac{2^\mu d^\nu t^\nu}{\lambda e^{dt} + 1}\right)^{\alpha} e^{cdXt} \varphi(Y, cdt) \Big(\frac{\lambda^d e^{cdt} + 1}{\lambda e^{ct} + 1}\Big). \tag{2.12}$$

Since expression (2.12) for $H(t)$ is symmetric in c and d, therefore we can expand $H(t)$ into series in two ways. First, we consider the following expansion:

$$H(t) = \frac{1}{c^{\nu\alpha} d^{\nu\alpha}} \left(\frac{2^\mu c^\nu t^\nu}{\lambda e^{ct} + 1}\right)^{\alpha} e^{cdxt} \varphi(dy, ct) \Big(\frac{\lambda^c e^{cdt} + 1}{\lambda e^{dt} + 1}\Big) \left(\frac{2^\mu d^\nu t^\nu}{\lambda e^{dt} + 1}\right)^{\alpha} e^{cdXt} \varphi(cY, dt) \Big(\frac{\lambda^d e^{cdt} + 1}{\lambda e^{ct} + 1}\Big). \tag{2.13}$$

Now, using the series expansions for $\Big(\frac{\lambda^c e^{cdt}+1}{\lambda e^{dt}+1}\Big)$ and $\Big(\frac{\lambda^d e^{cdt}+1}{\lambda e^{ct}+1}\Big)$ in the r.h.s. of equation (2.13), we find

$$H(t) = \frac{1}{c^{\nu\alpha} d^{\nu\alpha}} \left(\frac{2^\mu (ct)^\nu}{\lambda e^{ct} + 1}\right)^{\alpha} e^{cdxt} \varphi(dy, ct) \sum_{i=0}^{c-1} (-\lambda)^i e^{dti} \left(\frac{2^\mu (dt)^\nu}{\lambda e^{dt} + 1}\right)^{\alpha} e^{cdXt} \varphi(cY, dt) \sum_{j=0}^{d-1} (-\lambda)^j e^{ctj}. \tag{2.14}$$

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Combining the exponential terms in the above equation, we have

$$\begin{aligned} H(t) = \quad & \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{i=0}^{c-1}(-\lambda)^i\left(\tfrac{2^\mu (ct)^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{\left(dx+\frac{d}{c}i\right)ct}\varphi(dy,ct)\sum_{j=0}^{d-1}(-\lambda)^j\left(\tfrac{2^\mu (dt)^\nu}{\lambda e^{dt}+1}\right)^\alpha \\ & \times e^{\left(cX+\frac{c}{d}j\right)dt}\varphi(cY,dt), \end{aligned} \tag{2.15}$$

which on using equation (1.9) becomes

$$\begin{aligned} H(t) = \quad & \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\left(\sum_{i=0}^{c-1}(-\lambda)^i\sum_{n=0}^{\infty}{}_p\mathcal{F}_n^{(\alpha)}\left(dx+\tfrac{d}{c}i,dy;\lambda;\mu,\nu\right)\tfrac{(ct)^n}{n!}\right) \\ & \times\left(\sum_{j=0}^{d-1}(-\lambda)^j\sum_{n=0}^{\infty}{}_p\mathcal{F}_n^{(\alpha)}\left(cX+\tfrac{c}{d}j,cY;\lambda;\mu,\nu\right)\tfrac{(dt)^n}{n!}\right). \end{aligned} \tag{2.16}$$

Applying the Cauchy product rule in the r.h.s. of equation (2.16), we find

$$\begin{aligned} H(t) = \quad & \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^k d^{n-k}{}_p\mathcal{F}_k^{(\alpha)}\left(dx+\tfrac{d}{c}i,dy;\lambda;\mu,\nu\right) \\ & \times{}_p\mathcal{F}_{n-k}^{(\alpha)}\left(cX+\tfrac{c}{d}j,cY;\lambda;\mu,\nu\right)\tfrac{t^n}{n!}. \end{aligned} \tag{2.17}$$

Using the similar plan, we obtain the second expansion of $H(t)$ as:

$$\begin{aligned} H(t) \quad & = \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^k c^{n-k}{}_p\mathcal{F}_k^{(\alpha)}\left(cx+\tfrac{c}{d}i,cy;\lambda;\mu,\nu\right) \\ & \times{}_p\mathcal{F}_{n-k}^{(\alpha)}\left(dX+\tfrac{d}{c}j,dY;\lambda;\mu,\nu\right)\tfrac{t^n}{n!}. \end{aligned} \tag{2.18}$$

Equating the coefficients of same powers of t in r.h.s. of expansions (2.17) and (2.18), we are led to assertion (2.10).

Q.E.D.

Remark 2.1. In view of the special cases of the ATP $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ given in Table 2, the corresponding 2-variable Apostol type polynomials are defined in [12, p.1374(Table 2.1)]. Thus, by taking suitable values of the parameters in identities (2.1) and (2.10) of the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, the symmetry identities for the corresponding special cases can be obtained. We present the symmetry identities for these special cases in Table 3.

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Table 3. Symmetry identities for the polynomials ${}_p\mathfrak{B}_n^{(\alpha)}(x,y;\lambda)$, ${}_p\mathfrak{E}_n^{(\alpha)}(x,y;\lambda)$ and ${}_p\mathcal{G}_n^{(\alpha)}(x,y;\lambda)$.

S. No.	Values of the parameters	Relation between the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ and its special case	Name of the resultant special polynomials	Symmetry identities for the polynomials ${}_p\mathfrak{B}_n^{(\alpha)}(x,y;\lambda)$, ${}_p\mathfrak{E}_n^{(\alpha)}(x,y;\lambda)$ and ${}_p\mathcal{G}_n^{(\alpha)}(x,y;\lambda)$
I.	$\lambda \to -\lambda$ $\mu = 0$ $\nu = 1$	$(-1)^{\alpha}$ ${}_p\mathcal{F}_n^{(\alpha)}(x,y;-\lambda;0,1)$ $= {}_p\mathfrak{B}_n^{(\alpha)}(x,y;\lambda)$	2-variable Apostol-Bernoulli polynomials (2VABP) of order α	$\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{\nu+k}\,{}_p\mathfrak{B}_{n-k}^{(\alpha)}(dx,dy;\lambda)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(c-1;\lambda)\,{}_p\mathfrak{B}_{k-l}^{(\alpha-1)}(cX,cY;\lambda)$ $=\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{\nu+k}\,{}_p\mathfrak{B}_{n-k}^{(\alpha)}(cx,cy;\lambda)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(d-1;\lambda)\,{}_p\mathfrak{B}_{k-l}^{(\alpha-1)}(dX,dY;\lambda)$ and $\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(\lambda)^{i+j}c^k d^{n-k}\,{}_p\mathfrak{B}_k^{(\alpha)}\left(dx+\frac{d}{c}i,dy;\lambda\right){}_p\mathfrak{B}_{n-k}^{(\alpha)}\left(cX+\frac{c}{d}j,cY;\lambda\right)$ $\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(\lambda)^{i+j}d^k c^{n-k}\,{}_p\mathfrak{B}_k^{(\alpha)}\left(cx+\frac{c}{d}i,cy;\lambda\right){}_p\mathfrak{B}_{n-k}^{(\alpha)}\left(dX+\frac{d}{c}j,dY;\lambda\right)$
II.	$\mu = 1$ $\nu = 0$	${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;1,0)$ $= {}_p\mathfrak{E}_n^{(\alpha)}(x,y;\lambda)$	2-variable Apostol-Euler polynomials (2VAEP) of order α	$\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{\nu+k}\,{}_p\mathfrak{E}_{n-k}^{(\alpha)}(dx,dy;\lambda)\sum_{l=0}^{k}\binom{k}{l}\mathcal{M}_l(c-1;\lambda)\,{}_p\mathfrak{E}_{k-l}^{(\alpha-1)}(cX,cY;\lambda)$ $=\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{\nu+k}\,{}_p\mathfrak{E}_{n-k}^{(\alpha)}(cx,cy;\lambda)\sum_{l=0}^{k}\binom{k}{l}\mathcal{M}_l(d-1;\lambda)\,{}_p\mathfrak{E}_{k-l}^{(\alpha-1)}(dX,dY;\lambda)$ and $\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^k d^{n-k}\,{}_p\mathfrak{E}_k^{(\alpha)}\left(dx+\frac{d}{c}i,dy;\lambda\right){}_p\mathfrak{E}_{n-k}^{(\alpha)}\left(cX+\frac{c}{d}j,cY;\lambda\right)$ $\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^k c^{n-k}\,{}_p\mathfrak{E}_k^{(\alpha)}\left(cx+\frac{c}{d}i,cy;\lambda\right){}_p\mathfrak{E}_{n-k}^{(\alpha)}\left(dX+\frac{d}{c}j,dY;\lambda\right)$
III.	$\mu = 1$ $\nu = 1$	${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;1,1)$ $= {}_p\mathcal{G}_n^{(\alpha)}(x,y;\lambda)$	2-variable Apostol-Genocchi polynomials (2VAGP) of order α	$\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{\nu+k}\,{}_p\mathcal{G}_{n-k}^{(\alpha)}(dx,dy;\lambda)\sum_{l=0}^{k}\binom{k}{l}\mathcal{M}_l(c-1;\lambda)\,{}_p\mathcal{G}_{k-l}^{(\alpha-1)}(cX,cY;\lambda)$ $=\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{\nu+k}\,{}_p\mathcal{G}_{n-k}^{(\alpha)}(cx,cy;\lambda)\sum_{l=0}^{k}\binom{k}{l}\mathcal{M}_l(d-1;\lambda)\,{}_p\mathcal{G}_{k-l}^{(\alpha-1)}(dX,dY;\lambda)$ and $\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^k d^{n-k}\,{}_p\mathcal{G}_k^{(\alpha)}\left(dx+\frac{d}{c}i,dy;\lambda\right){}_p\mathcal{G}_{n-k}^{(\alpha)}\left(cX+\frac{c}{d}j,cY;\lambda\right)$ $\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^k c^{n-k}\,{}_p\mathcal{G}_k^{(\alpha)}\left(cx+\frac{c}{d}i,cy;\lambda\right){}_p\mathcal{G}_{n-k}^{(\alpha)}\left(dX+\frac{d}{c}j,dY;\lambda\right)$

Remark 2.2. We note that, for $\lambda = 1$ and in view of relations (1.7) and (1.12), the symmetry identities for the 2VABP ${}_p\mathfrak{B}_n^{(\alpha)}(x,y;\lambda)$, 2VAEP ${}_p\mathfrak{E}_n^{(\alpha)}(x,y;\lambda)$ and 2VAGP ${}_p\mathcal{G}_n^{(\alpha)}(x,y;\lambda)$ reduce to the corresponding identities for the 2-variable Bernoulli polynomials (2VBP) (of order α) ${}_pB_n^{(\alpha)}(x,y)$, 2-variable Euler polynomials (2VEP) (of order α) ${}_pE_n^{(\alpha)}(x,y)$ and 2-variable Genocchi polynomials (2VGP) (of order α) ${}_pG_n^{(\alpha)}(x,y)$, respectively.

Remark 2.3. Again, we note that for $\lambda = \alpha = 1$ and in view of relations (1.7), (1.8) and (1.12), the identities mentioned in Table 3 for the 2VABP ${}_p\mathfrak{B}_n^{(\alpha)}(x,y;\lambda)$, 2VAEP ${}_p\mathfrak{E}_n^{(\alpha)}(x,y;\lambda)$ and 2VAGP ${}_p\mathcal{G}_n^{(\alpha)}(x,y;\lambda)$ reduce to the corresponding identities for the 2-variable Bernoulli polynomials (2VBP) ${}_pB_n(x,y)$, 2-variable Euler polynomials (2VEP) ${}_pE_n(x,y)$ and 2-variable Genocchi polynomials (2VGP) ${}_pG_n(x,y)$.

In the next section, symmetry identities for certain members belonging to the family of 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ are derived.

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3 Examples

The members belonging to the 2VGP family are given in Table 1. It has been shown in [12], that corresponding to each member belonging to the family of 2VGP $p_n(x,y)$, there is a new special polynomial belonging to the 2VATP ${}_p\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ family.

In order to derive the symmetry identities for these members, we consider the following examples:

Example 3.1. By taking $\varphi(y,t)=e^{yt^m}$ (Table 1 (I)) in the l.h.s. of generating function (1.9), we get the following generating function for the 2-variable Gould-Hopper-Apostol type polynomials (2VGHATP) ${}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ [12, p.1375(3.1)]:

$$\left(\frac{2^\mu\ t^\nu}{\lambda e^t+1}\right)^\alpha e^{xt+yt^m}=\sum_{n=0}^{\infty}{}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)\frac{t^n}{n!}. \tag{3.1}$$

To derive the symmetry identities for the 2VGHATP ${}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, we prove the following results:

Theorem 3.1. For all integers $c,\ d>0$ and $n\geq 0,\ \alpha\geq 1,\ \lambda,\mu,\nu\in\mathbb{C}$, the following symmetry identity for the 2VGHATP ${}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ holds true:

$$\begin{aligned}&\textstyle\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{\nu+k}{}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}(dx,d^my;\lambda;\mu,\nu)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(c-1;-\lambda){}_{H^{(m)}}\mathcal{F}_{k-l}^{(\alpha-1)}(cX,c^mY;\lambda;\mu,\nu)\\ &=\textstyle\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{\nu+k}{}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}(cx,c^my;\lambda;\mu,\nu)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(d-1;-\lambda){}_{H^{(m)}}\mathcal{F}_{k-l}^{(\alpha-1)}(dX,d^mY;\lambda;\mu,\nu).\end{aligned} \tag{3.2}$$

Proof. Taking $\varphi(y,cdt)=e^{y(cdt)^m}$ and $\varphi(Y,cdt)=e^{Y(cdt)^m}$ in expression (2.2) of $G(t)$, we have

$$G_1(t):=\frac{2^{\mu(2\alpha-1)}t^{\nu(2\alpha-1)}e^{cdxt+y(cdt)^m}(\lambda e^{cdt}+1)e^{cdXt+Y(cdt)^m}}{(\lambda e^{ct}+1)^\alpha(\lambda e^{dt}+1)^\alpha}. \tag{3.3}$$

Rearranging the powers in expression (3.3), we find

$$G_1(t)=\frac{1}{c^{\nu\alpha}d^{\nu(\alpha-1)}}\left(\frac{2^\mu c^\nu t^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{cdxt+y(cdt)^m}\left(\frac{\lambda e^{cdt}+1}{\lambda e^{dt}+1}\right)\left(\frac{2^\mu d^\nu t^\nu}{\lambda e^{dt}+1}\right)^{\alpha-1}e^{cdXt+Y(cdt)^m}. \tag{3.4}$$

Since the expression (3.4) for $G_1(t)$ is symmetric in c and d. Therefore $G_1(t)$ can be expanded into series in two ways. Consider the expansion

$$G_1(t)=\frac{1}{c^{\nu\alpha}d^{\nu(\alpha-1)}}\left(\frac{2^\mu c^\nu t^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{dxct+d^my(ct)^m}\left(\frac{\lambda e^{cdt}+1}{\lambda e^{dt}+1}\right)\left(\frac{2^\mu d^\nu t^\nu}{\lambda e^{dt}+1}\right)^{\alpha-1}e^{cXdt+c^mY(dt)^m}, \tag{3.5}$$

which in view of equations (1.11) and (3.1) becomes

$$\begin{aligned}G_1(t)=&\ \tfrac{1}{c^{\nu\alpha}d^{\nu(\alpha-1)}}\left(\textstyle\sum_{n=0}^{\infty}{}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(dx,d^my;\lambda;\mu,\nu)\frac{(ct)^n}{n!}\right)\left(\sum_{l=0}^{\infty}\mathcal{S}_l(c-1;-\lambda)\frac{(dt)^l}{l!}\right)\\ &\times\left(\textstyle\sum_{k=0}^{\infty}{}_{H^{(m)}}\mathcal{F}_k^{(\alpha-1)}(cX,c^mY;\lambda;\mu,\nu)\frac{(dt)^k}{k!}\right).\end{aligned} \tag{3.6}$$

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Again, using [21, p.890 Corollary 2], we find

$$G_1(t) = \frac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{\nu+k}{}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}(dx,d^my;\lambda;\mu,\nu)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(c-1;-\lambda)$$
$$\times{}_{H^{(m)}}\mathcal{F}_{k-l}^{(\alpha-1)}(cX,c^mY;\lambda;\mu,\nu)\Big)\frac{t^n}{n!}. \tag{3.7}$$

Using a similar plan, we have

$$G_1(t) = \frac{1}{d^{\nu\alpha}c^{\nu\alpha}}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{\nu+k}{}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}(cx,c^my;\lambda;\mu,\nu)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(d-1;-\lambda)$$
$$\times{}_{H^{(m)}}\mathcal{F}_{k-l}^{(\alpha-1)}(dX,d^mY;\lambda;\mu,\nu)\Big)\frac{t^n}{n!}. \tag{3.8}$$

Equating the coefficients of same powers of t in r.h.s. of expansions (3.7) and (3.8), we are led to assertion (3.2). Q.E.D.

Theorem 3.2. For each pair of positive integers $c,\ d$ and for all integers $n\geq 0,\ \alpha\geq 1,\ \lambda,\mu,\nu\in\mathbb{C}$, the following symmetry identity for the 2VGHATP ${}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ holds true:

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^kd^{n-k}{}_{H^{(m)}}\mathcal{F}_k^{(\alpha)}\Big(dx+\tfrac{d}{c}i,d^my;\lambda;\mu,\nu\Big){}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}\Big(cX+\tfrac{c}{d}j,c^mY;\lambda;\mu,\nu\Big)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^kc^{n-k}{}_{H^{(m)}}\mathcal{F}_k^{(\alpha)}\Big(cx+\tfrac{c}{d}i,c^my;\lambda;\mu,\nu\Big){}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}\Big(dX+\tfrac{d}{c}j,d^mY;\lambda;\mu,\nu\Big). \tag{3.9}$$

Proof. Taking $\varphi(y,cdt)=e^{y(cdt)^m}$ and $\varphi(Y,cdt)=e^{Y(cdt)^m}$ in expression (2.11) of $H(t)$, we have

$$H_1(t):=\frac{2^{2\mu\alpha}t^{2\nu\alpha}e^{cdxt}e^{y(cdt)^m}(\lambda^ce^{cdt}+1)(\lambda^de^{cdt}+1)e^{cdXt}e^{Y(cdt)^m}}{(\lambda e^{ct}+1)^{\alpha+1}(\lambda e^{dt}+1)^{\alpha+1}}. \tag{3.10}$$

Rearranging the powers in expression (3.10), we find

$$H_1(t)=\frac{1}{c^{\nu\alpha}d^{\nu\alpha}}\left(\frac{2^\mu c^\nu t^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{cdxt+y(cdt)^m}\left(\frac{\lambda^ce^{cdt}+1}{\lambda e^{dt}+1}\right)\left(\frac{2^\mu d^\nu t^\nu}{\lambda e^{dt}+1}\right)^\alpha e^{cdXt+Y(cdt)^m}\left(\frac{\lambda^de^{cdt}+1}{\lambda e^{ct}+1}\right). \tag{3.11}$$

Since the expression (3.11) for $H_1(t)$ is symmetric in c and d. Therefore, $H_1(t)$ can be expanded into series in two ways. Consider the expansion

$$H_1(t)=\frac{1}{c^{\nu\alpha}d^{\nu\alpha}}\left(\frac{2^\mu c^\nu t^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{dxct+d^my(ct)^m}\left(\frac{\lambda^ce^{cdt}+1}{\lambda e^{dt}+1}\right)\left(\frac{2^\mu d^\nu t^\nu}{\lambda e^{dt}+1}\right)^\alpha e^{dXct+d^mY(ct)^m}\left(\frac{\lambda^de^{cdt}+1}{\lambda e^{ct}+1}\right). \tag{3.12}$$

Now, using the series expansions for $\left(\frac{\lambda^ce^{cdt}+1}{\lambda e^{dt}+1}\right)$ and $\left(\frac{\lambda^de^{cdt}+1}{\lambda e^{ct}+1}\right)$ in the r.h.s. of above equation, we find

$$H_1(t)=\frac{1}{c^{\nu\alpha}d^{\nu\alpha}}\left(\frac{2^\mu(ct)^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{dxct+d^my(ct)^m}\sum_{i=0}^{c-1}(-\lambda)^ie^{dti}\left(\frac{2^\mu(dt)^\nu}{\lambda e^{dt}+1}\right)^\alpha e^{dXct+d^mY(ct)^m}\sum_{j=0}^{d-1}(-\lambda)^je^{ctj}. \tag{3.13}$$

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Combining the exponential terms in the above equation, we have

$$H_1(t)=\frac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{i=0}^{c-1}(-\lambda)^i\left(\frac{2^\mu(ct)^\nu}{\lambda e^{ct}+1}\right)^\alpha e^{\left(dx+\frac{d}{c}i\right)ct+d^my(ct)^m}\sum_{j=0}^{d-1}(-\lambda)^j\left(\frac{2^\mu(dt)^\nu}{\lambda e^{dt}+1}\right)^\alpha e^{\left(cX+\frac{c}{d}j\right)dt+c^mY(dt)^m}, \tag{3.14}$$

which on using equation (3.1) becomes

$$\begin{aligned}H_1(t)= &\ \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\left(\sum_{i=0}^{c-1}(-\lambda)^i\sum_{n=0}^{\infty}{}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}\left(dx+\tfrac{d}{c}i,d^my;\lambda;\mu,\nu\right)\tfrac{(ct)^n}{n!}\right)\\ &\times\left(\sum_{j=0}^{d-1}(-\lambda)^j\sum_{n=0}^{\infty}{}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}\left(cX+\tfrac{c}{d}j,c^mY;\lambda;\mu,\nu\right)\tfrac{(dt)^n}{n!}\right).\end{aligned} \tag{3.15}$$

Applying the Cauchy product rule in the r.h.s. of equation (3.15), we find

$$\begin{aligned}H_1(t)= &\ \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^kd^{n-k}{}_{H^{(m)}}\mathcal{F}_k^{(\alpha)}\left(dx+\tfrac{d}{c}i,d^my;\lambda;\mu,\nu\right)\\ &\times{}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}\left(cX+\tfrac{c}{d}j,c^mY;\lambda;\mu,\nu\right)\tfrac{t^n}{n!}.\end{aligned} \tag{3.16}$$

Using a similar plan, we obtain the second expansion of $H_1(t)$ as:

$$\begin{aligned}H_1(t)= &\ \tfrac{1}{c^{\nu\alpha}d^{\nu\alpha}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^kc^{n-k}{}_{H^{(m)}}\mathcal{F}_k^{(\alpha)}\left(cx+\tfrac{c}{d}i,c^my;\lambda;\mu,\nu\right)\\ &\times{}_{H^{(m)}}\mathcal{F}_{n-k}^{(\alpha)}\left(dX+\tfrac{d}{c}j,d^mY;\lambda;\mu,\nu\right)\tfrac{t^n}{n!}.\end{aligned} \tag{3.17}$$

Equating the coefficients of same powers of t in r.h.s. of expansions (3.16) and (3.17), we are led to assertion (3.9).

Q.E.D.

Remark 3.1. By taking suitable values of the parameters in identities (3.2) and (3.9) and in view of relations given in Table 3 (I-III), the symmetry identities for the special cases of ${}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ can be obtained.

Remark 3.2. We know that for $m=2$, the GHP $H_n^{(m)}(x,y)$ reduce to 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP) $H_n(x,y)$ (Table 1 (II)). Therefore, taking $m=2$ in symmetry identities (3.2) and (3.9) of the 2VGHATP ${}_{H^{(m)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, we find the symmetry identities for the 2-variable Hermite-Apostol type polynomials (2VHATP) ${}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$. These identities can also be viewed as particular cases of the identities obtained in [17]. Further, taking suitable values of the parameters in identities of the 2VHATP ${}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ and in view of relations given in Table 3 (I-III), we obtain the symmetry identities for the special cases of the 2VHATP ${}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, see for example [17, 19].

Remark 3.3. We know that for $x\to 2x$ and $y=-1$, the 2VHKdFP $H_n(x,y)$ reduce to the classical Hermite polynomials $H_n(x)$ [1]. Therefore, taking $x\to 2x$ and $y=-1$ in identities of the 2VHATP ${}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, we obtain the symmetry identities for the Hermite-Apostol type polynomials,

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which may be denoted by ${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$. Further, taking suitable values of the parameters in identities of the ${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$, the corresponding symmetry identities for the special cases of ${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ can be obtained, see for example [18].

Example 3.2. By taking $\varphi(y,t)=C_0(-yt^m)$ (Table 1 (III)) in the l.h.s. of generating function (1.9), we get the following generating function for the 2-variable generalized Laguerre-Apostol type polynomials (2VGLATP) ${}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$ [12, p.1377(3.4)]:

$$\Big(\frac{2^{\mu}\ t^{\nu}}{\lambda e^t+1}\Big)^{\alpha}e^{xt}C_0(-yt^m)=\sum_{n=0}^{\infty}{}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)\frac{t^n}{n!}. \tag{3.18}$$

We derive the symmetry identities for the 2VGLATP ${}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$ by proving the following results:

Theorem 3.3. For all integers $c,\ d>0$ and $n\geq 0,\ \alpha\geq 1,\ \lambda,\mu,\nu\in\mathbb{C}$, the following symmetry identity for the 2VGLATP ${}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$ holds true:

$$\begin{aligned}&\textstyle\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{\nu+k}{}_{mL}\mathcal{F}_{n-k}^{(\alpha)}(d^my,dx;\lambda;\mu,\nu)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(c-1;-\lambda)_{mL}\mathcal{F}_{k-l}^{(\alpha-1)}(c^mY,cX;\lambda;\mu,\nu)\\ &=\textstyle\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{\nu+k}{}_{mL}\mathcal{F}_{n-k}^{(\alpha)}(c^my,cx;\lambda;\mu,\nu)\sum_{l=0}^{k}\binom{k}{l}\mathcal{S}_l(d-1;-\lambda)_{mL}\mathcal{F}_{k-l}^{(\alpha-1)}(d^mY,dX;\lambda;\mu,\nu).\end{aligned} \tag{3.19}$$

Proof. Taking $\varphi(y,cdt)=C_0(-y(cdt)^m)$ and $\varphi(Y,cdt)=C_0(-Y(cdt)^m)$ in expression (2.2) of $G(t)$, we have

$$G_2(t):=\frac{2^{\mu(2\alpha-1)}t^{\nu(2\alpha-1)}e^{cdxt}C_0(-y(cdt)^m)(\lambda e^{cdt}+1)e^{cdXt}C_0(-Y(cdt)^m)}{(\lambda e^{ct}+1)^{\alpha}(\lambda e^{dt}+1)^{\alpha}}. \tag{3.20}$$

Now, rearranging the powers in expression (3.20) of $G_2(t)$ and using the fact that $G_2(t)$ is symmetric in c and d, we can obtain two expansions of $G_2(t)$ as in Theorem 3.1. Finally, equating the coefficients of same powers of t in these expansions, we get assertion (3.19). Q.E.D.

Theorem 3.4. For each pair of positive integers $c,\ d$ and for all integers $n\geq 0,\ \alpha\geq 1,\ \lambda,\mu,\nu\in\mathbb{C}$, the following symmetry identity for the 2VGLATP ${}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$ holds true:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^kd^{n-k}{}_{mL}\mathcal{F}_k^{(\alpha)}\Big(d^my,dx+\tfrac{d}{c}i;\lambda;\mu,\nu\Big){}_{mL}\mathcal{F}_{n-k}^{(\alpha)}\Big(c^mY,cX+\tfrac{c}{d}j;\lambda;\mu,\nu\Big)\\ &=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^kc^{n-k}{}_{mL}\mathcal{F}_k^{(\alpha)}\Big(c^my,cx+\tfrac{c}{d}i;\lambda;\mu,\nu\Big){}_{mL}\mathcal{F}_{n-k}^{(\alpha)}\Big(d^mY,dX+\tfrac{d}{c}j;\lambda;\mu,\nu\Big).\end{aligned} \tag{3.21}$$

Proof. Taking $\varphi(y,cdt)=C_0(-y(cdt)^m)$ and $\varphi(Y,cdt)=C_0(-Y(cdt)^m)$ in expression (2.11) of $H(t)$, we have

$$H_2(t):=\frac{2^{2\mu\alpha}t^{2\nu\alpha}e^{cdxt}C_0(-y(cdt)^m)(\lambda^ce^{cdt}+1)(\lambda^de^{cdt}+1)e^{cdXt}C_0(-Y(cdt)^m)}{(\lambda e^{ct}+1)^{\alpha+1}(\lambda e^{dt}+1)^{\alpha+1}}. \tag{3.22}$$

Q.E.D.

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Now, rearranging the powers in expression (3.22) of $H_2(t)$ and using the fact that $H_2(t)$ is symmetric in c and d, we can obtain two expansions of $H_2(t)$ as in Theorem 3.2. Finally, equating the coefficients of same powers of t in these expansions, we get assertion (3.21).

Remark 3.4. By taking suitable values of the parameters in identities (3.19) and (3.21) and in view of relations Table 3 (I-III), the symmetry identities for the special cases of the ${}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$ can be obtained.

Remark 3.5. We know that for $m=1$ and $y \to -y$, the 2VGLP ${}_mL_n(y,x)$ reduce to the 2-variable Laguerre polynomials $L_n(y,x)$ (Table 1 (IV)). Therefore, taking $m=1$ and $y \to -y$ in symmetry identities (3.19) and (3.21) of the 2VGLATP ${}_{mL}\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$, we obtain the symmetry identities for the 2-variable Laguerre-Apostol type polynomials (2VLATP) ${}_L\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$.

Further, taking suitable values of the parameters in identities for the ${}_L\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$ and in view of relations given in Table 3 (I-III), we obtain the symmetry identities for the special cases of the 2VLATP ${}_L\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$.

Remark 3.6. We know that for $x=1$, the 2VLP $L_n(y,x)$ reduce to the classical Laguerre polynomials $L_n(y)$ [1]. Therefore, taking $x=1$ in identities for the ${}_L\mathcal{F}_n^{(\alpha)}(y,x;\lambda;\mu,\nu)$, we obtain the corresponding identities for the Laguerre-Apostol type polynomials, which may be denoted by ${}_L\mathcal{F}_n^{(\alpha)}(y;\lambda;\mu,\nu)$.

Further, taking suitable values of the parameters in identities of the ${}_L\mathcal{F}_n^{(\alpha)}(y;\lambda;\mu,\nu)$, the corresponding symmetry identities for the special cases of ${}_L\mathcal{F}_n^{(\alpha)}(y;\lambda;\mu,\nu)$ can be obtained.

Example 3.3. By taking $\varphi(y,t)=\frac{1}{1-yt^r}$ and $\varphi(Y,t)=\frac{1}{1-Yt^r}$ (Table 1 (V)) in the l.h.s. of generating function (1.9), we get the following generating function for the 2-variable truncated exponential-Apostol type polynomials (2VTEATP) ${}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ [12, p.1377(3.7)]:

$$\left(\frac{2^\mu\ t^\nu}{\lambda e^t+1}\right)^\alpha\left(\frac{e^{xt}}{1-yt^r}\right)=\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)\frac{t^n}{n!}. \tag{3.23}$$

We derive the symmetry identities of 2VTEATP ${}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ by proving the following results:

Theorem 3.5. For all integers $c,\ d>0$ and $n\geq 0,\ \alpha\geq 1,\ \lambda,\mu,\nu\in\mathbb{C}$, the following symmetry identity for the 2VTEATP ${}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ holds true:

$$\begin{aligned}&\textstyle\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}c^{n-k}d^{\nu+k}{}_{e^{(r)}}\mathcal{F}_{n-k}^{(\alpha)}(dx,d^ry;\lambda;\mu,\nu)S_l(c-1;-\lambda){}_{e^{(r)}}\mathcal{F}_{k-l}^{(\alpha-1)}(cX,c^rY;\lambda;\mu,\nu)\\&=\textstyle\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}d^{n-k}c^{\nu+k}{}_{e^{(r)}}\mathcal{F}_{n-k}^{(\alpha)}(cx,c^ry;\lambda;\mu,\nu)S_l(d-1;-\lambda){}_{e^{(r)}}\mathcal{F}_{k-l}^{(\alpha-1)}(dX,d^rY;\lambda;\mu,\nu).\end{aligned} \tag{3.24}$$

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Proof. Taking $\varphi(y,cdt)=\frac{1}{1-y(cdt)^r}$ and $\varphi(Y,cdt)=\frac{1}{1-Y(cdt)^r}$ in expression (2.2) of $G(t)$, we have

$$G_3(t):=\frac{2^{\mu(2\alpha-1)}t^{\nu(2\alpha-1)}e^{cdxt}\frac{1}{1-y(cdt)^r}(\lambda e^{cdt}+1)e^{cdXt}\frac{1}{1-Y(cdt)^r}}{(\lambda e^{ct}+1)^{\alpha}(\lambda e^{dt}+1)^{\alpha}}. \tag{3.25}$$

Now, rearranging the powers in expression (3.25) of $G_3(t)$ and using the fact that $G_3(t)$ is symmetric in c and d, we can obtain two expansions of $G_3(t)$ as in Theorem 3.1. Finally, equating the coefficients of same powers of t in these expansions, we get assertion (3.24). Q.E.D.

Theorem 3.6. For each pair of positive integers $c,\ d$ and for all integers $n\geq 0,\ \alpha\geq 1,\ \lambda,\mu,\nu\in\mathbb{C}$, the following symmetry identity for the 2VTEATP ${}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ holds true:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^kd^{n-k}{}_{e^{(r)}}\mathcal{F}_k^{(\alpha)}\Big(dx+\tfrac{d}{c}i,d^ry;\lambda;\mu,\nu\Big){}_{e^{(r)}}\mathcal{F}_{n-k}^{(\alpha)}\Big(cX+\tfrac{c}{d}j,c^rY;\lambda;\mu,\nu\Big)\\&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^kc^{n-k}{}_{e^{(r)}}\mathcal{F}_k^{(\alpha)}\Big(cx+\tfrac{c}{d}i,c^ry;\lambda;\mu,\nu\Big){}_{e^{(r)}}\mathcal{F}_{n-k}^{(\alpha)}\Big(dX+\tfrac{d}{c}j,d^rY;\lambda;\mu,\nu\Big).\end{aligned} \tag{3.26}$$

Proof. Taking $\varphi(y,cdt)=\frac{1}{1-y(cdt)^r}$ and $\varphi(Y,cdt)=\frac{1}{1-Y(cdt)^r}$ in expression (2.11) of $H(t)$, we have

$$H_3(t):=\frac{2^{2\mu\alpha}t^{2\nu\alpha}e^{cdxt}\frac{1}{1-y(cdt)^r}(\lambda^c e^{cdt}+1)(\lambda^d e^{cdt}+1)e^{cdXt}\frac{1}{1-Y(cdt)^r}}{(\lambda e^{ct}+1)^{\alpha+1}(\lambda e^{dt}+1)^{\alpha+1}}. \tag{3.27}$$

Now, rearranging the powers in expression (3.27) of $H_3(t)$ and using the fact that $H_3(t)$ is symmetric in c and d, we can obtain two expansions of $H_3(t)$ as in Theorem 3.2. Finally, equating the coefficients of same powers of t in these expansions, we get assertion (3.26).

Q.E.D.

Remark 3.7. By taking suitable values of the parameters in identities (3.24) and (3.26) and in view of relations Table 3 (I-III), the symmetry identities for the special cases of the ${}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ can be obtained.

Remark 3.8. We know that for $r=2$, the 2VTEP $e^{(r)}(x,y)$ of order r reduce to the 2VTEP ${}_{[2]}e_n(x,y)$ (Table 1 (VI)). Therefore, taking $r=2$ in symmetry identities (3.24) and (3.26) of the 2VTEATP ${}_{e^{(r)}}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, we obtain the symmetry identities for the ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$. Further, taking suitable values of the parameters in identities of the ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$ and in view of relations given in Table 3 (I-III), we obtain the symmetry identities for the special cases of ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$.

Remark 3.9. We know that for $y=1$, the 2VTEP ${}_{[2]}e_n(x,y)$ reduce to the truncated exponential polynomials ${}_{[2]}e_n(x)$ [5]. Therefore, taking $y=1$ in the identities of the ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu,\nu)$, we obtain the corresponding identities for the truncated exponential Apostol type polynomials, which may be denoted by ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$. Further, taking suitable values of the parameters in

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identities of the ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$, the corresponding symmetry identities for the special cases of ${}_{[2]e}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ can be obtained.

The Bernoulli, Euler and Genocchi numbers B_n, E_n and G_n have deep connections with number theory and occur in combinatorics. These numbers appear as special values of the Bernoulli, Euler and Genocchi polynomials $B_n(x)$, $E_n(x)$ and $G_n(x)$, respectively given as

$$B_n := B_n(0) := B_n^{(1)}(0);\ E_n := E_n(0) := E_n^{(1)}(0);\ G_n := G_n(0) := G_n^{(1)}(0), \tag{3.28}$$

respectively.

The Apostol type numbers of order α denoted by $\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)$ are defined by the generating function

$$\Big(\frac{2^\mu\ t^\nu}{\lambda e^t+1}\Big)^\alpha = \sum_{n=0}^{\infty}\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)\frac{t^n}{n!},\ |t|<|\log(-\lambda)|. \tag{3.29}$$

Consequently, from equations (1.3) and (3.29), we have

$$\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu) := \mathcal{F}_n^{(\alpha)}(0;\lambda;\mu,\nu). \tag{3.30}$$

In view of the special cases mentioned in Table 2 and using the fact that

$$\mathfrak{B}_n^{(\alpha)}(0;\lambda) = \mathfrak{B}_n^{(\alpha)}(\lambda);\ \mathfrak{E}_n^{(\alpha)}(0;\lambda) = \mathfrak{E}_n^{(\alpha)}(\lambda);\ \mathcal{G}_n^{(\alpha)}(0;\lambda) = \mathcal{G}_n^{(\alpha)}(\lambda), \tag{3.31}$$

where $\mathfrak{B}_n^{(\alpha)}(\lambda)$, $\mathfrak{E}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n^{(\alpha)}(\lambda)$ denote the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi numbers of order α, respectively, we have the following special cases of $\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)$ [12]:

$$(-1)^\alpha\mathcal{F}_n^{(\alpha)}(-\lambda;0,1) = \mathfrak{B}_n^{(\alpha)}(\lambda);\ \mathcal{F}_n^{(\alpha)}(\lambda;1,0) = \mathfrak{E}_n^{(\alpha)}(\lambda);\ \mathcal{F}_n^{(\alpha)}(\lambda;1,1) = \mathcal{G}_n^{(\alpha)}(\lambda). \tag{3.32}$$

The Hermite Apostol type numbers (HATN) of order α, ${}_H\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)$ are defined as [12]:

$${}_H\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{F}_{n-k}^{(\alpha)}(\lambda;\mu,\nu)H_k, \tag{3.33}$$

where $H_k := H_k(0)$ are the Hermite numbers. We also note that [12, p.1380(4.12)]:

$${}_H\mathcal{F}_n^{(\alpha)}(0;\lambda;\mu,\nu) := {}_H\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu). \tag{3.34}$$

The importance of the numbers related to the polynomials provides motivation to establish the results for the HATN ${}_H\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)$ of order α and the numbers related to other special polynomials considered in this paper. This aspect may be taken in further investigation.

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